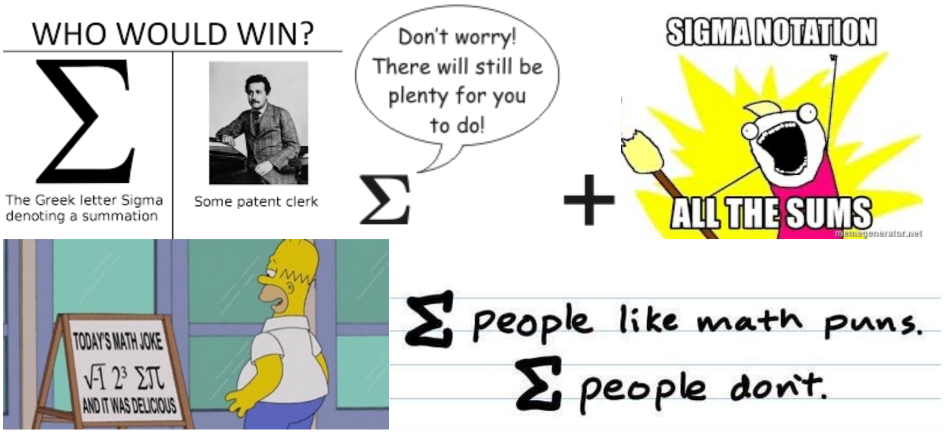
Sequences & Series: Sigma Notation



This sheet assumes that you already know how to deal with arithmetic and geometric series.

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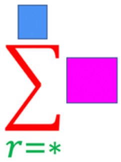
This is a long worksheet to cater for students that want extra practice. If you want a shortcut, but still be sure to cover one of each type then follow the pink highlighted questions.

- **Higher level students** should be able to do all questions up until diamond to be sure to get a 7. The challenge section is there for anyone brave enough ☺
- **Standard level students** should be able to do questions 1-36 and 41-42 to be sure to get a 7

Make sure you have covered the sequences and series: arithmetic and geometric worksheet first!

Reminders

Sigma notation $\sum$ is another way to say **sum**



Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like. To do this we replace the pink box with first 3 or 4 values of * (i.e. we just plug in the values to the expression in the pink box)

Step 2: Simplify each term and decide what type of sequence we have (the 3 or 4 terms written out above is enough for us to see the pattern).

- **arithmetic** - if the sequence has a common difference (adds or subtracts the same number each time
- **geometric -** if the sequence has a common ratio (multiplies or divides by the same number each time)
- **periodic** – if the sequence repeats after a certain number of terms

For example,

- let's say we have the general sequence a, b, c, d

  If $b - a = c - b \Longrightarrow$ series is arithmetic

  If $\frac{b}{a} = \frac{c}{b} \Longrightarrow$series is geometric

- If we have the general sequence a, b, c, a, b, c then we said the sequence is periodic of order 3 (since it repeats every 3 terms)

Step 3: The sigma notation is related to the sum formula that you should be familiar with for geometric and arithmetic series. We can use the s_n formula for arithmetic or geometric.

- $\sum_{r=1}^{n} \dots = S_n$
- $\sum_{r=m}^{n} \dots$
  The sum must start from 1 to use s_n formula $\Longrightarrow$ $\sum_{r=m}^{n} \dots = \sum_{r=1}^{n} \dots - \sum_{r=1}^{m-1} \dots = S_n - S_{m-1}$
  Note: This has $n - m + 1$ terms
- $\sum_{r=1}^{\infty} \dots = S_\infty$
- $\sum_{r=m}^{\infty} \dots = S_\infty - S_{m-1}$ or we can just find the first term for that sum, call it a and find $\frac{a}{1-r}$

Examples:

$$\sum_{r=1}^{r=6} (r+1)$$

Let's colour code to explain

$$\sum_{r=1}^{r=6} (r+1)$$

In English, this says replace every r starting from 1 in the expression $(r+1)$ and go all the way to 6.
We add ($\sum$ means add) all these terms found.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=6} (r+1)$$

To do this we replace $(r+1)$ with the values of r
We know r starts at 1 and ends at 6
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(1+1) + (2+1) + (3+1) + (4+1) + (5+1) + (6+1)$$
$$= 2 + 3 + 4 + 5 + 6 + 7$$

Step 2: Decide what type of series we have
Here we keep adding 1 each time so we have an arithmetic sequence with $a = 2$ and $d = 1$

Step 3: Find the sum
Way 1: Since we only have a few terms we can find the sum easily: $2 + 3 + 4 + 5 + 6 + 7 = 27$
Way 2: use the s_n formula for an arithmetic series with $a = 2$ and $d = 1$. We would do this way if we had more terms we only have 6 terms here

$$S_6 = \frac{6}{2}[2(2) + (6-1)(1)] = 27$$

What happens if we have more terms?

$$\sum_{r=1}^{r=50}(4r+1)$$

In English, this says replace every r starting from 1 in the expression $(r+1)$ and go all the way to 50.
We add ($\sum$ means add) all these terms found.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=50}(4r+1)$$

To do this we replace $(4r+1)$ with the values of r
We know r starts at 1 and ends at 50

$$(4+1)+(8+1)+(12+1)+\cdots+(50+1)$$

Simplifying gives the sequence

5 +9+13 +...+201

This is arithmetic with a=5 and d=4 and n is 50 (50 terms). If you didn't realise there are 50 terms, you could set $u_n = 201$ using the formula and solve for n

Using s_n formula

$$s_{50} = \frac{50}{2}[2(5)+(50-1)(4)] = 25(10+196)=5150$$

What happens if our series doesn't start from 1?

$$\sum_{r=5}^{r=80} 2(3^{r+1})$$

Way 1:	Way 2: Force the sum to start from 1
The sum says replace every r starting from 5 $2(3^6)+2(3^7)+2(3^8)+\cdots$ This is geometric since we are multiplying by 3 each time $r=\frac{2(3^7)}{2(3^6)}=3$ $a=2(3^6)=1458$ From 5 to 80 is 74 terms $s_{74}=\frac{1457(1-3^{74})}{1-3}$	$=\sum_{r=1}^{80}2(3^{r+1})-\sum_{r=1}^{4}2(3^{r+1})$ This says replace every r starting from 1 in each sequence $[2(3^2)+2(3^3)+2(3^4)+\cdots]-[2(3^2)+2(3^3)+2(3^4)]$ Each series is geometric with $r=\frac{2(3^3)}{2(3^2)}=3$ $a=2(3^2)=18$ $=S_{80}-S_4$ $=\frac{18(1-3^{80})}{1-3}-\frac{18(1-3^4)}{1-3}$

1 Bronze

1) Find the following by writing out a suitable series $\sum_{r=1}^{r=6}(r+1)$

2) Write down all the terms of the following
   i. $\sum_{k=1}^{6}(2k+1)$
   ii. $\sum_{k=3}^{7}k^2$
   iii. $\sum_{k=4}^{8}k(2^{2k-1})$
   iv. $\sum_{k=100}^{100}(3k-7)$

3) Calculate $\sum_{r=1}^{r=5}3r$

4) Calculate $\sum_{r=0}^{r=5}r(r+1)$

5) Calculate $\sum_{r=1}^{r=20}(4r+1)$

6) Calculate $\sum_{r=1}^{r=20}(5r-2)$

7) Find the value of $\sum_{r=1}^{42}(5r+3)$

2 Silver

8) Calculate $\sum_{n=1}^{n=20} 3n$ and $\sum_{n=21}^{n=100} 3n$

9) Calculate $\sum_{r=10}^{r=30}(7+2r)$

10) Calculate $\sum_{r=3}^{r=6}(2^r-1)$

11) Calculate $\sum_{r=-1}^{r=4}(1.5^r)$

12) Expand $\sum_{r=4}^{7} 2^r$ as the sum of four terms
   i. Find the value of $\sum_{r=4}^{30} 2^r$
   ii. Explain why $\sum_{r=4}^{\infty} 2^r$ cannot be evaluated

13) The r^{th} term of an arithmetic series is ($2r$ −5)
   i. Write down the first three terms of the series
   ii. State the value of the common difference
   iii. Show that $\sum_{r=1}^{n}(2r-5) = n(n-4)$

14) Prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

15) Find the value of $\sum_{r=1}^{6} 10 \times \left(\frac{2}{3}\right)^{r-1}$

16) Find the value of $\sum_{r=1}^{\infty} 10 \times \left(\frac{2}{3}\right)^{r-1}$

17) Find the value of $\sum_{r=7}^{\infty} 10 \times \left(\frac{2}{3}\right)^{r-1}$

18) Find the value of $\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^{r}$

19) Let $s_n = \sum_{r=1}^{n}(2r-3)$
   i. Write down the first three terms of this series
   ii. Find S_{50}
   iii. Find n such that $S_n = 575$

20) Consider the geometric series $S_n = \sum_{r=1}^{n} 3 \times 5^{r-1}$
Given that $S_m = 7324218$, work out the value of m

21) For what value of n does $\sum_{r=1}^{n}(5r+3)$ first exceed 1000

22) Given $\sum_{r=1}^{n}(100-3r) < 0$. Find the least value of the integer n

23) For what value of n would $\sum_{r=1}^{n}(100-4r)=0$

24) Given that $\sum_{r=1}^{n} a_r = 12+4n^2$.

i. Find the value of $\sum_{r=1}^{5} a_r$ and the value of a_6

ii. Given that $\sum_{r=0}^{\infty}\frac{a}{4^r}=16$. Find the value of the constant a

25) The n[th] term of an arithmetic progression is denoted by u_n, and given by $u_n = 2n+7$
Determine the value N given that $\sum_{n=1}^{N} u_n = 2100$

26) Show that $\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n cos\,(180n)° = \frac{9}{28}$

27) Find the value of k if $\sum_{r=1}^{\infty} k\left(\frac{1}{3}\right)^r = 7$

28) Rewrite 4+7+10+…+31 using sigma notation

29) Rewrite 40+36+32+…+0 using sigma notation

30) Rewrite 3+$3^2+3^3+3^4+\cdots+3^8$ using sigma notation

31) Rewrite the multiples of 6 less than 100 using sigma notation

3 Gold

32) The third and the sixth term of a geometric progression is 27 and 8, respectively.

Show clearly that $\sum_{r=6}^{\infty} u_r = 24$, Where u_r is the r[th] term of the progression.

33) A geometric series, u_n has second term 375 and fifth term 81. Find the sum to infinity and hence the value of $\sum_{n=6}^{\infty} u_n$

3.1 With Logs

34) Show that the terms of $\sum_{r=1}^{m} \ln 3^r$ are in arithmetic series
   i. Find the sum of the first 20 terms in this series
   ii. Hence, show that $\sum_{r=1}^{2m} \ln 3^r = \ln 3\,(2m^2 + m)$

35) Find $\sum_{r=1}^{50} \ln(2^r)$, giving the answer in the form aln2, where a∈ $\mathbb{Q}$

36) The first three terms of a geometric sequence are $lnx^{16}, lnx^{8}, lnx^{4}$ for x>0.
   i. Find the common ratio
   ii. Solve $\sum_{k=1}^{\infty} 2^{5-k} lnx = 64$

4 Diamond

4.1 With Logs

37) Given that p is a positive constant,

i. Show that $\sum_{n=1}^{11} \ln(p^n) = klnp$,where k is a constant to be found

ii. Show that $\sum_{n=1}^{11} \ln(8p^n) = 33\ln(2p^2)$

iii. Hence find the set of values p for which $\sum_{n=1}^{11} \ln(8p^n) < 0$ giving your answer in set notation

38) Find $\sum_{n=1}^{15} {a_n}^2$ where $a_n = lnx^n$

39) Show that $\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right)$=2

40) Show that $S = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2}$

4.2 Two Series

41) Show that $\sum_{r=1}^{16}(3 + 5r + 2^r)$=131,798

42) Find $\sum_{n=0}^{\infty} \frac{2^n+4^n}{6^n}$

43) $\sum_{r=1}^{10} a + (r-1)d = \sum_{r=11}^{14} a + (r-1)d$. Show that $d = 6a$

44) An arithmetic series has first term a and common difference d. The sum of the first 15 terms is 1320 and the fifth term is 76

i. Find a and d

Given that $13\left(\sum_{n=1}^{15} u_n - \sum_{n=1}^{k} u_n\right) = 9\sum_{n=1}^{k}\ u_k$

ii. find the value of S_k

iii. Hence find the value of k

5 Challenges

5.1 Arithmetic

45) The first term of an arithmetic series is a and the common different is d
The 25th term is 100.
The 5th term is 8 times larger than the 35th term of the series.
    i. Find the values of a and the value of d
    ii. Determine how many terms of the series as positive
The sum of the first n terms of the series is denoted by S_n
    iii. Calculate the maximum value of S_n

46) $\sum_{n=1}^{20}(2r+x)=280$. Find the value of x that satisfies the equation.

47) The nth term of an arithmetic series is given by

$$u_n=\frac{5}{2}(5n+28)$$

The kth term of the series is 370.
    i. Find the value of k
    ii. Evaluate the sum $\sum_{n=1}^{k} u_n$

48) Find the value of the constant p, so that $\sum_{n=1}^{20}(25+np)=80$

49) A sequence is defined as $u_{r+1}=u_r-3$, $u_{1=117}, n\geq 1$
Solve the equation $\sum_{r=1}^{n} u_r=0$

50) Find in simplified for the terms of n, the value of

$$\sum_{r=1}^{2n}(3r-2)(-1)^r$$

51) The rth term of an arithmetic progression is given by $u_r=120-3r$
Determine the value N given that $\sum_{r=N}^{3N} u_r=444$

52) An arithmetic progression has first term -10 and common difference 4. The nth term of the progression is denoted by u_n. Determine the value of k given that

$$\sum_{n=1}^{2k} u_n-\sum_{n=1}^{k} u_n=1728$$

53) The sum of the first 25 terms of an arithmetic series if 1050 and its 25th term is 72.
i. Find the first term and the common difference of the series.
The nth term of the series of denoted by u_n
ii. Given further that

$$117\left[\sum_{n=1}^{25} u_n-\sum_{n=1}^{k} u_n\right]=233\sum_{n=1}^{k} u_n$$

Determine the value of k

54) The r[th] term of an arithmetic progression is denoted by u_r and satisfies $u_r = 4r - 7$
Solve the simultaneous equations

$$\sum_{r=K+1}^{N} u_r - \sum_{r=1}^{K} , u_r = 400$$

$$u_N - u_K = 40$$

5.2 Geometric

55) The sum of the geometric series is 2187.
The (k−1)[th] and k[th] term of the same series are 96 and 64, respectively. Determine the value of

$$\sum_{n=k+1}^{\infty} u_n$$

Where u_n, is the n[th] term of the series. Determine the value of

56) It is given that

$$\sum_{r=1}^{n} u_r = 128 - 2^{7-n}$$

Where u_r is the r[th] term of the geometric progression.

i. Find the sum of the first 8 terms of the progression
ii. Determine the value of u_8
iii. Find the common ratio

57) Determine the value n, given by $\sum_{r=1}^{n} 2^{2r-1} = 43690$

58) The n[th] term of a geometric series is denoted by u_n. It is given that

$$u_1 = 1458 \text{ and } u_6 = 6.$$

Evaluate: $\sum_{n=7}^{\infty} u_n$

59) A family of an infinite geometric series S_k has a first term $\frac{k-1}{k!}$ and common ratio $\frac{1}{k}$, where k = 3,4,5,6,...,99,100.
Find the value of $\frac{10^4}{100!} + \sum_{k=3}^{100} [[(k-1)(k-2) - 1]S_k]$.

60) By showing a detailed method, sum the following series. Find the value of:

$$\sum_{r=0}^{9} [(r+1) \times 11^r \times 10^{9-r}]$$

You may leave your answer in index form

61) Show that the following equation how only one real solution

$$27n = 4\sum_{r=2}^{\infty} (1+n)^{-r}$$

62) Evaluate the following expression $\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{1}{2^{m+n}}\right)$. Detailed working must be shown.

63) Evaluate the following expression

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{1}{3^{m+n}}\right)$$

Detailed working must be shown.

64) It is given that the following series converges to the limit L.

$$\sum_{r=1}^{\infty} \left[\frac{2x-1}{x+2}\right]^r$$

Determine with full justification the range of possible values of L.

65) Solve the following simultaneous equations

$$2\sum_{r=0}^{\infty} [\log_2 a]^r = \sum_{k=1}^{\infty} [1+b]^{-k} \quad \text{and} \quad 2\sum_{k=1}^{1} [1+b]^{-k} - \sum_{r=0}^{1} [\log_2 a]^r = \frac{7}{5}$$

Write your answers in index form where appropriate

66) The (k−1)th and kth terms of the convergent geometric progression are 108 and 81. Determine the value of

$$\sum_{n=k+1}^{\infty} u_n$$

Where u_n is the nth term of the series

67) It is given that

$$S_n = \sum_{k=1}^{n} \sum_{r=1}^{k} (2^r)$$

Show that

$$S_n = 2^{n+2} - 2n - 4$$

68) Evaluate showing clearly your method

$$S_n = \sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}}$$

69) It is given that 0 < r < 1, 0 < R < 1 and r < 2R. It is further given that

$$\sum_{n=0}^{\infty} R^n = \left(\sum_{n=0}^{\infty} r^n\right)^2$$

Show clearly that

$$\sum_{n=0}^{\infty} \left(\frac{r}{2R}\right)^n = \frac{2(2-r)}{3-2r}$$